THE DAVENPORT-HALBERSTAM THEOREM FOR MÖBIUS FUNCTION

STEVE FAN

ABSTRACT. In Chapter 29 of Davenport's classic book [1], it is shown that given any $A > 0$ we have

$$\sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\psi(x;q,a)-\frac{x}{\phi(q)}\right|^2 \ll xQ\log x$$

uniformly for all $x(\log x)^{-A}\le Q\le x$, where

$$\psi(x;q,a) := \sum_{\substack{n\le x\\ n\equiv a\ (\mathrm{mod}\ q)}}\Lambda(n)$$

and $\Lambda(n)$ is the von Mangolt function. This result is known as the Davenport-Halberstam theorem. In this short note we present a simple proof of the following folklore version for the Möbius function $\mu(n)$ (for instance, see [3, Theorem 2]): for any given $A > 0$,

$$\sum_{q\le Q}\sum_{a=1}^{q}|M(x;q,a)|^2 = \frac{6}{\pi^2}xQ + O\left(x^2(\log x)^{-A}\right)$$

holds uniformly for all $x(\log x)^{-A}\le Q\le x$, where

$$M(x;q,a) := \sum_{\substack{n\le x\\ n\equiv a\ (\mathrm{mod}\ q)}}\mu(n).$$

The author learned this result from a Number Theory Web Seminar talk given by Robert C. Vaughan in 2022.

In Chapter 29 of Davenport's classic book [1], it is shown that given any $A > 0$ we have

$$\sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\psi(x;q,a)-\frac{x}{\phi(q)}\right|^2 \ll xQ\log x$$

uniformly for all $x(\log x)^{-A}\le Q\le x$, where

$$\psi(x;q,a) := \sum_{\substack{n\le x\\ n\equiv a\ (\mathrm{mod}\ q)}}\Lambda(n)$$

and $\Lambda(n)$ is the von Mangolt function. This result is known as the Davenport-Halberstam theorem. Subsequent improvements have been obtained by Montgomery [4], who shows that the left-hand side above is

$$Qx\log x + O(Qx\log(2x/Q)) + O\left(x^2(\log x)^{-A}\right)$$

in the stated range, and by Hooley [2], who derives the following asymptotic formula for the left-hand side above with an explicit second-order term:

$$Qx\log Q - cQX + O\left(Q^{5/4}x^{3/4} + x^2(\log x)^{-A}\right)$$

for some constant $c \in \mathbb{R}$. In this short note we present a simple proof of the following folklore version for the Möbius function $\mu(n)$ (for instance, see [3, Theorem 2]).

Theorem. *Fixing an arbitrary $A > 0$ we have*

$$\sum_{q\le Q}\sum_{a=1}^{q}|M(x;q,a)|^2 = \frac{6}{\pi^2}xQ + O\left(x^2(\log x)^{-A}\right)$$

uniformly for all $x(\log x)^{-A} \le Q \le x$, where

$$M(x;q,a) := \sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} \mu(n).$$

Proof. Set $Q_0 := x(\log x)^{-A}$. Applying the arithmetic large sieve [5, Theorem 4.13] to the sequence

$$a_n := \begin{cases} \mu(n) & \text{if } n \le x \text{ with } n \equiv a\,(\mathrm{mod}\,q),\\ 0 & \text{otherwise,}\end{cases}$$

we have

$$|M(x;q,a)|^2 \le \frac{x+q^2}{L_q}\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}}\mu(n)^2,$$

where

$$L_q := \sum_{m\le q}\mu(m)^2\prod_{p\mid m}\frac{w(p)}{p-w(p)}$$

and

$$w(p) := \#\{h\in\mathbb{F}_p\colon a_n = 0 \text{ for all } n\le x \text{ with } n\equiv a\,(\mathrm{mod}\,q) \text{ and } n\equiv h\,(\mathrm{mod}\,p)\} < p.$$

Since $w(p) = p-1$ for all $p \mid q$, it follows that

$$L_q \ge \sum_{m\le q}\mu(m)^2\prod_{p\mid m,\, p\mid q}(p-1) = \sum_{m\le q}\mu(m)^2\varphi((m,q)) \ge \sum_{m\le q}\mu(m)^2 \gg q,$$

where φ is the Euler totient function. Hence

$$\sum_{q\le Q_0}\sum_{a=1}^{q}|M(x;q,a)|^2 \ll \sum_{q\le Q_0}\frac{x+q^2}{q}\sum_{n\le q}\mu(n)^2 \ll \left(x\log Q_0 + Q_0^2\right)x \ll x^2(\log x)^{-A}. \qquad (1)$$

On the other hand, we write

$$\sum_{Q_0<q\le Q}\sum_{a=1}^{q}|M(x;q,a)|^2 = (\lfloor Q\rfloor - \lfloor Q_0\rfloor)\sum_{n\le x}\mu(n)^2 + 2\sum_{Q_0<q\le Q}\sum_{\substack{m<n\le x\\ m\equiv n\,(\mathrm{mod}\,q)}}\mu(m)\mu(n). \qquad (2)$$

Note that

$$\sum_{Q_0<q\le Q}\sum_{\substack{m<n\le x\\ m\equiv n \,(\mathrm{mod}\, q)}}\mu(m)\mu(n)=\sum_{r<x/Q_0}\sum_{m<x-Q_0r}\mu(m)\sum_{\substack{m+Q_0r<n\le\min(x,m+Qr)\\ n\equiv m\,(\mathrm{mod}\, r)}}\mu(n). \tag{3}$$

Now we appeal to the following version of the Siegel-Walfisz theorem: for any fixed $B>0$ we have

$$M(x;q,a)\ll x\exp(-c(B)\sqrt{\log x})$$

uniformly for all $q\le(\log x)^B$ and $1\le a\le q$, where $c(B)>0$ depends only on B. Since $x/Q_0=(\log x)^A$, the right-hand side of (3) is

$$\begin{aligned}&\ll x\exp(-c(A)\sqrt{\log x})\sum_{r<x/Q_0}\sum_{m<x-Q_0r}|\mu(m)|\\ &\ll x(\log x)^A\exp(-c(A)\sqrt{\log x})\sum_{n\le x}|\mu(m)|\\ &\ll x^2(\log x)^{-A}.\end{aligned}$$

Inserting this estimate into (2) we obtain

$$\sum_{Q_0<q\le Q}\sum_{a=1}^{q}|M(x;q,a)|^2=\frac{6}{\pi^2}xQ+O\left(x^2(\log x)^{-A}\right).$$

Combining this with (1) yields the desired asymptotic formula. □

References

[1] H. Davenport, *Multiplicative Number Theory*, 3rd. ed., Grad. Texts in Math., vol. 74, Springer-Verlag, New York, 2000. Revised and with a preface by H. L. Montgomery.

[2] C. Hooley, *On the Barban-Davenport-Halberstam theorem I*, J. Reine Angew. Math., **274/5** (1975), 206–223.

[3] C. Hooley, *On the Barban-Davenport-Halberstam theorem III*, J. Lond. Math. Soc. **10** (2) (1975), 249–256.

[4] H. L. Montgomery, *Primes in arithmetic progressions*, Michigan Math. J., **17** (1970), 33–39.

[5] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, 3rd. ed., Grad. Stud. Math., vol. 163, Amer. Math. Soc., Providence, RI., 2015.

Department of Mathematics, Dartmouth College, Hanover, NH 03755, USA

Email address: steve.fan.gr@dartmouth.edu